

Wormhole Cosmic Censorship

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We analyze the properties of a Kerr-like wormhole supported by phantom matter, which is an exact solution of the Einstein-phantom field equations with two parameters: mass and scalar field charge. The solution has a naked ring singularity which is unreachable to null geodesics falling freely from outside. Similarly to Roger Penrose's cosmic censorship, here we show that a naked singularity can be fully protected by the intrinsic properties of the wormhole's spacetime.

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Singularities are a natural ingredient of the solutions of the Einstein field equations [1]. Most of the exact solutions of these equations representing local objects contain real singularities in their space-time structure. Penrose conjectured that these singularities should be protected by an event horizon which avoids that we can see the singularity of any space-time.

Recently, phantom matter has emerged as a promising candidate to be the dark energy in the universe[5]. At the local level, phantom matter may be the matter source of wormholes[2–4], so that wormholes are becoming again one of the most mysterious and interesting solutions of Einstein's equations [6].

In this letter, we study in some detail the physical properties of the Kerr-like wormhole found in [7], which is in fact an exact solution of the Einstein's equations sourced by a phantom scalar field. As we shall show, the relevant feature of the solution is that its internal ring singularity is completely unreachable because of the protection provided by the (special) wormhole's throat.

To begin with, we write the line element describing the spacetime around the wormholes in Boyer-Linquist coordinates as

$$ds^2 = -f dt^2 + \frac{K}{f} dl^2 + \frac{\Delta_1}{f} [K d\theta^2 + \sin^2 \theta d\varphi^2] , \quad (1)$$

where $K = \Delta/\Delta_1$, and

$$\Delta = (l - l_1)^2 + (l_0^2 - l_1^2) \cos^2 \theta , \quad (2a)$$

$$\Delta_1 = (l - l_1)^2 + (l_0^2 - l_1^2) , \quad (2b)$$

being l_1 and l_0 two parameters with units of distance such that $l_0^2 > l_1^2 > 0$. Also, $f = e^{-\lambda}$ and

$$\lambda = \frac{k_1}{2\Delta} \cos \theta , \quad (3)$$

where k_1 is a parameter with units of angular momentum. The meaning of function λ can be seen from the fact that the line element (1) is an exact solution of the Einstein's equations, $R_{\mu\nu} = -8\pi G \Phi_\mu \Phi_\nu$, where

$$\Phi = \frac{1}{\sqrt{16\pi G}} \lambda , \quad (4)$$

is a phantom-type scalar field; for more details about the solution see [7].

For large values, $|l| \gg l_0, l_1$, we have $\lambda \rightarrow 0$, $f \rightarrow 1$ and $\Delta, \Delta_1 \rightarrow l^2$; thus, the line element (1) is asymptotically flat,

$$ds^2 \rightarrow -dt^2 + dl^2 + l^2 (d\theta^2 + \sin^2 \theta d\varphi^2) . \quad (5)$$

The restriction $l_0^2 > l_1^2 > 0$ for the parameters l_0 and l_1 assures that $\Delta_1 > 0$ everywhere, but that is not the case for Δ , which can be zero at different points of the spacetime. Thus, metric (1) has a naked singularity surface determined by the condition $\Delta = 0$. That is, there is a naked ring singularity for $\theta = \pi/2$ and $l = l_1$.

To verify that we have encountered a true singularity, we should take a look at the invariants of the metric. For our case, straightforward calculations show that the invariants can be generally written as

$$\text{Invariants} = \frac{F}{8k_1^2 \Delta^{\alpha_1} \Delta_1^{\alpha_2} (k_1^2 e^{2\lambda})^{\alpha_3}} , \quad (6)$$

where F is a complicated function free of singularities that takes different forms for each invariant of the metric, and α_1 , α_2 , and α_3 are coefficients whose exact value depend upon the chosen invariant.

In Fig. 1 we show the behavior of the (invariant) Ricci scalar of metric (1) for different values of the coordinate θ . As expected, the Ricci scalar diverges at $l = l_1$ on the equator, which is the position of the naked singularity.

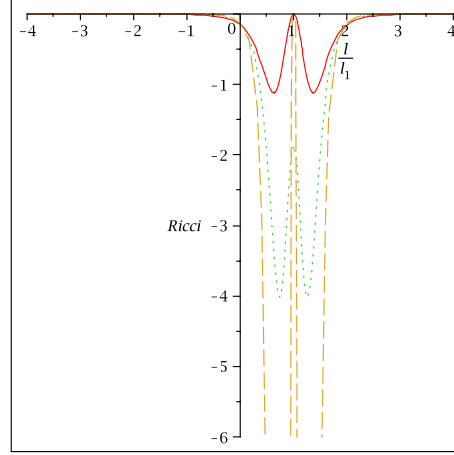


FIG. 1: The Ricci scalar for the metric (1) with $l_1 = 1$, $l_0 = 1.1$, $k_1 = 1$, for different values of θ ; on the north pole $\theta = 0$ (solid line), for $\theta = \pi/4$ (dot line) and for $\theta \lesssim \pi/2$, (dashed line). The behavior depicted is typical of the Ricci scalar, even though the chosen values are arbitrary.

Some other geometrical properties of the wormhole are discussed in turn. The quantity inside the squared brackets in Eq. (1) can be interpreted as a solid angle modified by function K ; notice that $K \sim 1$ everywhere except in the singularity, where $K \sim 0$. The modification of the solid angle indicates that the spacetime is axially symmetric rather than spherically symmetric. Nevertheless, the modification of the solid angle is small except at a sphere of radius l_1 .

On the other hand, the function $r^2 \equiv \Delta_1/f$ plays the role of a squared radial coordinate. To have a better visualization of geodesics, let us consider the following pseudo-cartesian coordinates: $x = l \cos \theta$, and $z = l \sin \theta$. We can observe in Fig.2 that the radial coordinate is a complicated function, which is nonetheless well behaved everywhere, except for the locations of the naked singularities.

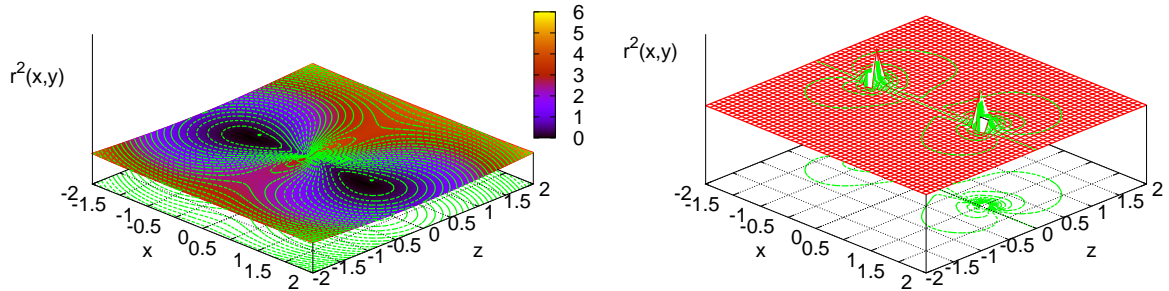


FIG. 2: The squared radial coordinate $r^2 = \Delta_1/f$, for the values $l_1 = 1$, $l_0 = 2$, and $k_1 = 1$, in terms of the pseudo-cartesian coordinates x, z defined in the text; shown are the full 3d graph (top), and a projection on the xz plane (bottom). Notice that function $r^2(x, y)$ is well behaved everywhere, except for the points at which naked singularities are located. The radius $r(x, z)$ is in general non-zero on the upper half, $z > 0$, of the xz plane. In particular, the radius is non-zero at the origin of coordinates, which points out the existence of a throat at this point.

Let us now study the null geodesics of test particles freely falling into the wormhole. This will give us more elements to understand the properties of the wormhole's spacetime[8] and the role of the naked singularities. To begin with,

we write the Lagrangian for geodesics,

$$\mathcal{L} = -f\dot{t}^2 + \frac{K}{f}\dot{l}^2 + \frac{\Delta_1}{f} \left[K\dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2 \right], \quad (7)$$

where a dot denotes derivative with respect to an affine parameter. Because the line element does not depend upon t and φ explicitly, their respective momenta are conserved,

$$p_t = f\dot{t} = \text{const.}, \quad p_\varphi = \frac{\Delta_1}{f} \sin^2 \theta \dot{\varphi} = \text{const.}$$

Without any loss of generality, we set $p_\varphi = 0$ and $\varphi = 0$ hereafter. For the remaining variables, the Euler-Lagrange equations read

$$\dot{l} = \Delta_1 \exp \left[-\frac{k_1 \cos \theta}{2\Delta} \right] \frac{p_l}{\Delta}, \quad \dot{\theta} = \exp \left[-\frac{k_1 \cos \theta}{2\Delta} \right] \frac{p_\theta}{\Delta}, \quad (8a)$$

$$\dot{p}_l = -\frac{k_1(l-l_1) \cos \theta}{2f\Delta^2} p_t^2 + \frac{f(l-l_1)\Delta_1}{2\Delta^3} [2\Delta - k_1 \cos \theta] \left(p_l^2 + \frac{p_\theta^2}{\Delta_1} \right) + \frac{f(l-l_1)}{\Delta} p_l^2, \quad (8b)$$

$$\dot{p}_\theta = -\frac{k_1 \Delta_- \sin \theta}{4f\Delta^2} p_t^2 - \frac{f\Delta_1 \sin \theta}{4\Delta^3} [4(l_0^2 - l_1^2)\Delta \cos \theta + k_1 \Delta_-] \left(p_l^2 + \frac{p_\theta^2}{\Delta_1} \right), \quad (8c)$$

where $\Delta_- = (l-l_1)^2 - (l_0^2 - l_1^2) \cos^2 \theta$. There is also the constant of motion that comes from the Lagrangian itself,

$$\mathcal{L} = -\frac{p_t^2}{f} + \frac{f}{K} \left(p_l^2 + \frac{p_\theta^2}{\Delta_1} \right) = \text{const.} \quad (9)$$

Eq. (9) will be used to monitor the error generated in the numerical solution of Eqs (8). The system (8b) and (8c) has an exact solution when $\theta = \frac{\pi}{2}$ but does not satisfy the condition set for null geodesics in (9) even when $p_\phi \neq 0$. And the solutions thus obtained has no physical value.

Because the wormhole's spacetime is not spherically symmetric, the angular momentum p_θ is not conserved along the geodesics. As our main interest is to search for geodesics that can travel through the wormholes throat, we will set the initial values $p_t = 1$ and $p_\theta = 0$ for all geodesics in our numerical experiments.

That is, initially, all geodesics are freely falling in the radial direction only. A non-zero angular momentum will just make it more difficult for a geodesic to reach the throat.

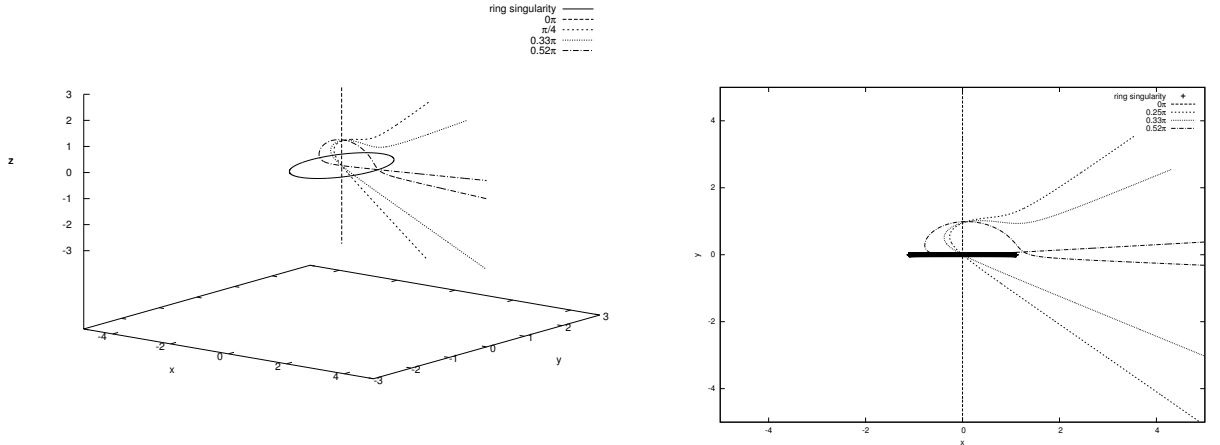


FIG. 3: Different null geodesics that fall into the wormhole from different initial values of θ ; the trajectories are numerical solutions of the geodesic equations (8). The right panel is the plane projection of the trajectories from the left panel. The ring singularity is also shown as long dashed lines. Geodesics are able to avoid the naked singularities, which are then *hidden* from the view of outside observers by the geodesic structure of the wormhole's spacetime.

The resulting trajectories are shown in Fig. 3. In general terms, we can see that null geodesics are able to avoid the naked singularities of spacetime. There even seems to be a kind of *closed* null surface surrounding the naked singularities.

Geodesics in Fig. 3 go from their positive values of the radial coordinate ℓ , to the negative values of ℓ on the other side of the throat, we interpret the latter as the part of the geodesics that is able to *traverse* through the wormhole's throat located at the origin. Furthermore, we have found that geodesics for some initial conditions cannot enter the wormhole's throat and are scattered off.

The naked singularities appear isolated in Fig. 3 from the space explored by null geodesics. This can be intuitively understood in terms of the conservation equation (9). For null geodesics we can write

$$p_t^2 + \frac{p_\theta^2}{\Delta_1} = \frac{K}{f^2} p_t^2. \quad (10)$$

As we have seen before, function f diverges at the location of the naked singularities, see for instance Fig. 2. As any geodesic trajectory approaches a singularity, the rhs of Eq. (10) diverges, and then the only possibility to keep the balance in the equality is to have large values of both the radial and the angular momenta.

In particular, angular momentum is not conserved, and then the geodesic trajectory is scattered off the singularity points. The closer the trajectory is to a singularity, the larger the angular momentum it acquires, the larger the diversion the geodesic takes.

In other words, the naked singularities are protected by an angular potential barrier, rather than by an event horizon. This results from the combined effect of the divergence nature of the singularities and the breaking of spherical symmetry in the spacetime points around them.

Our case resembles the case of an event horizon surrounding a singularity at the centre of a black hole. According to the famous Roger Penrose's cosmic censorship, the Universe needs the existence of event horizons to avoid any undesirable effects coming from a naked singularity.

The case of the wormhole we have studied throughout this letter maybe equally interesting. According to our solution, it is also possible to protect a ring singularity with a wormhole's throat. This may indicate a generalization of Penrose's cosmic censorship: it can be the case that naked singularities with a more involved configuration may find the formation of a wormhole around them more convenient than the appearance of a event horizon.

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